

THE METHOD OF ZONES FOR THE CALCULATION
OF TEMPERATURE DISTRIBUTION

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THE METHOD OF ZONES FOR THE CALCULATION OF TEMPERATURE DISTRIBUTION

SUMMARY

The method of zones is an improved method for obtaining approximate solutions to certain partial differential equations. Its application to heat transfer problems is discussed in detail. Within a zone the temperature is assumed to vary parabolically with the space coordinates, and mean temperatures throughout the volume and over the boundaries are calculated. The higher order of approximation of the method permits a complicated system to be subdivided into fewer parts than is necessary when conventional methods are used.

The heat flow equation is integrated over the volume of the zone to give an instantaneous heat balance equation which involves the fluxes over the boundaries of the zone and the rate of change of the mean temperature of the zone. Approximate formulas, which are based on the parabolic assumption, are derived expressing the fluxes in terms of the mean temperatures of the zone and its boundaries. These formulas are worked out for zones of various shapes. The boundary conditions are also integrated to obtain equations for relating the temperatures in zones that are joined together.

The instantaneous heat balance equations are integrated numerically with respect to time by means of a general linear two-point integration formula involving an integration parameter. Rules for choosing this parameter to insure stability and accuracy are given. A rule is also given for selecting the time increment.

Three simple examples are given to illustrate the application and accuracy of the method of zones. A number of criteria which enable zone sizes to be chosen properly in practical applications are developed. These criteria are derived by comparing calculations made by the method of zones with exact results in simple cases.

The modified Gauss-Seidel procedure for solving the difference equations at each time step is briefly discussed, and experience with the convergence of this procedure in the present connection is reviewed.

Finally, checking procedures, based on the principles of reciprocity and the conservation of energy, are introduced. These provide an initial check on the consistency of the input data and a running check on the solution of the problem.

INTRODUCTION

The method of zones was developed for calculating the temperature distributions in the components of complicated artificial satellites.

Conventional numerical methods^(1,2) for determining the temperature distribution in a component, such as a rod or a plate, involve the establishment of a suitably fine mesh in the object which defines points at which temperatures are to be evaluated. The calculations are then made by replacing the differential equation by a set of difference equations. This is equivalent to making the assumption that the temperature distribution between points of the mesh is linear in the space coordinates.

In the zone method, on the other hand, we divide the component into zones of suitable size and assume that the temperature within the zone varies parabolically with the space coordinates. Furthermore we evaluate mean temperatures averaged over the volume and over the boundaries of the zone instead of temperatures at particular mesh points. This method lends itself very naturally to the solution of boundary value problems and the higher order approximation used permits the size of our zones to be larger than the mesh size used in conventional methods. We will present rules to help to determine the subdivision of the system into zones of appropriate size. The use of larger zones greatly decreases the labor of calculating thermal conductances and view areas for radiation, and also decreases the magnitude of the machine calculations.

We will show later that the use of mean temperatures leads to greater accuracy in problems involving radiation and in transient problems. The use of mean temperatures and the parabolic assumption permit the explicit calculation of fluxes over boundaries which enables the interaction between zones to be accurately accounted for.

The method of zones is not restricted to problems in heat flow alone, but could be used in obtaining approximate solutions to many other partial differential equations, such as the biharmonic equation of the theory of elasticity.

INSTANTANEOUS HEAT BALANCE

The heat flow equation in a zone is

$$-KV^2T - p + C_p \frac{\partial T}{\partial t} = 0 \quad (1)$$

where K is the thermal conductivity, watts $\text{cm}^{-1} (\text{deg K})^{-1}$ which we assume to be constant.

T is the temperature, deg K.

p is the power input per unit volume, watts cm^{-3} .

C_p is the heat capacity per unit volume, joule $\text{cm}^{-3} (\text{deg K})^{-1}$.

t is the time, seconds.

In order to obtain solutions of Equation 1 suitable boundary conditions and an initial temperature distribution must be prescribed. In general

the boundary condition is given as a relationship between the temperature and flux at each point of the boundary

$$f(T, K \frac{\partial T}{\partial n}) = 0 \quad (2)$$

where n represents the outward directed normal and f is an arbitrary function. For example, in the case of a free boundary interchanging radiation with an environment at temperature T_0 , the boundary condition becomes

$$K \frac{\partial T}{\partial n} + \epsilon \sigma T^4 - \epsilon \sigma T_0^4 = 0 \quad (3)$$

where ϵ is the emissivity of the surface and σ is the Stefan-Boltzmann constant, watts cm^{-2} $(\text{deg K})^{-4}$. On the other hand, when two bodies are in contact, they share the common boundary conditions

$$q - K \frac{\partial T}{\partial n} - K' \frac{\partial T'}{\partial n'} = 0 \quad (4)$$

$$T - T' = 0 \quad (5)$$

where the prime refers to the second body and q is the heat generated at the interface, watts cm^{-2} . Equations 4 and 5 merely require continuity of the temperature and the heat flux over the boundary.

In order to obtain the overall heat balance for a zone, we integrate Equation 1 over the volume, V , of the zone. On applying Gauss' theorem to first term of Equation, and interchanging the order of differentiation and integration in the last term, we find

$$-K \int_S \frac{\partial T}{\partial n} dS - \int_V \rho dV + C_p \frac{d}{dt} \int_V T dV = 0 \quad (6)$$

where S is the surface of the zone. It will be noticed that

$$\int_V T \, dV = VT_m \quad (7)$$

where T_m is the mean temperature over the volume. Since VT_m does not vary with the space coordinates, the partial derivative with respect to time in Equation 1 becomes a total derivative. Also

$$\int_V p \, dV = P \quad (8)$$

where P is the total power dissipated in the volume, watts. If we divide the boundary of the zone into a number of faces S_1, S_2, S_3, \dots , the first term of Equation 6 is seen to be

$$-K \int_S \frac{\partial T}{\partial n} \, dS = Q_1 + Q_2 + Q_3 + \dots \quad (9)$$

$$\text{where } Q_1 = -K \int_{S_1} \frac{\partial T}{\partial n_1} \, dS_1 \text{ etc.} \quad (10)$$

Thus Q_1, Q_2, Q_3, \dots , are the total outward fluxes crossing S_1, S_2, S_3, \dots , in watts. On substituting the expressions (7), (8), and (9) into Equation 6 we find

$$Q_1 + Q_2 + Q_3 + \dots - P + CpV \frac{dT_m}{dt} = 0 \quad (11)$$

The heat balance equation (11) is one of the basic equations of the method of zones. This equation is exact, and it should be pointed out that if,

for example, the center temperature of the zone were used in place of the mean temperature, an error would, in general, be incurred.

Next we obtain the heat balance for a particular face of a zone by integrating the boundary conditions over a surface, say S_1 . For example, Equation 3 may be integrated to give

$$-Q_1 + \epsilon \sigma \int_{S_1} T^4 dS_1 - S_1 \epsilon \sigma T_o^4 = 0. \quad (12)$$

On account of the non-linear character of Equation 12, we make the substitution

$$T = T_1 + \theta$$

where T_1 is the mean temperature of S_1 and θ is the deviation therefrom.

Then the integral in (12) becomes

$$\int_{S_1} T^4 dS_1 = S_1 T_1^4 \left(1 + \frac{6}{S_1} \int_{S_1} \left(\frac{\theta}{T_1} \right)^2 dS_1 + \dots \right) \quad (13)$$

The term in θ is exactly zero because of the definition of T_1 as the mean temperature of S_1

$$T_1 = \frac{1}{S_1} \int_{S_1} T dS_1 = T_1 + \frac{1}{S_1} \int_{S_1} \theta dS_1 \quad (14)$$

The higher order terms in Equation 13 can be neglected if

$$\frac{6}{S_1} \int_{S_1} \left(\frac{\theta}{T_1} \right)^2 dS_1 \ll 1. \quad (15)$$

In the method of zones, this term is neglected so that Equation 12 becomes

$$- Q_1 + S_1 \epsilon \sigma T_1^4 - S_1 \epsilon \sigma T_o^4 = 0 \quad (16)$$

and Inequality 15 provides one criterion for choosing the zone size.

If the face is rectangular and θ varies linearly over S_1 , then

$$\frac{6}{S_1} \int_S \left(\frac{\theta}{T_1} \right)^2 dS_1 \leq \left(\frac{\theta_{\max}}{T_1} \right)^2 \quad (17)$$

so that, for example, if θ_{\max} is 20% of T_1 in degrees Kelvin, a 4% error in the radiation flux will be made by neglecting the term in θ^2 .

Another example of the integration of zone boundary conditions is obtained from Equations 4 and 5. These become

$$Q + Q_1 + Q'_1 = 0 \quad (18)$$

$$T_1 - T'_1 = 0 \quad (19)$$

where Q is the total heat generated over S_1

T_1 is the mean temperature over S_1 , as before

Q_1 is the total heat flux crossing S_1 out of the first body,
as defined in Equation 10

Q'_1 is the total heat flux crossing S_1 out of the second body.

DIFFERENCE EQUATIONS (SPACE)

In order to convert Equations 11, 16, 18, and 19 into difference equations in the space coordinates, it is necessary to express Q_1 , Q_2 , Q_3 , etc., in terms of the mean temperatures T_m , T_1 , T_2 , T_3 , etc. To do this we assume that the temperature can be instantaneously approximated by a particular function of the space coordinates.

Rectilinear Solid

In the case of a rectilinear solid of sides a , b , and c as illustrated in Figure 1, we assume that

$$T(x,y,z,t) = \sum_{i=0}^2 \sum_{j=0}^2 \sum_{k=0}^2 a_{ijk}(t) x^i y^j z^k. \quad (20)$$

It will be noted that $T(x,y,z,t)$ is a quadratic function of each of the space variables considered separately and that the coefficients are, in general, functions of the time. Thus, Equation 20 can be written in the equivalent forms

$$T(x,y,z,t) = f_0(y,z,t) + x f_1(y,z,t) + x^2 f_2(y,z,t) \quad (21)$$

$$T(x,y,z,t) = g_0(x,z,t) + y g_1(x,z,t) + y^2 g_2(x,z,t) \quad (22)$$

$$T(x,y,z,t) = h_0(x,y,t) + z h_1(x,y,t) + z^2 h_2(x,y,t) \quad (23)$$

As shown in Figure 1, Q_1 and Q_2 are the total outward fluxes leaving the surfaces normal to the x-axis. The temperatures T_1 and T_2 are the mean temperatures of these faces, and T_m is the mean over the whole volume. The expressions for Q_1 and Q_2 in terms of T_1 , T_2 , and T_m can now be found by the use of Equation 21. First we calculate Q_1 and Q_2 from Equation 10

$$Q_1 = K \int_0^b \int_0^c \left(\frac{\partial T}{\partial x} \right)_{x=0} dydz = KbcF_1(t) \quad (24)$$

where

$$F_1(t) = \frac{1}{bc} \int_0^b \int_0^c f_1(y,z,t) dydz \quad (25)$$

$$Q_2 = -K \int_0^b \int_0^c \left(\frac{\partial T}{\partial x} \right)_{x=a} dydz = -Kbc[F_1(t) + 2\epsilon F_2(t)] \quad (26)$$

where

$$F_2(t) = \frac{1}{bc} \int_0^b \int_0^c f_2(y,z,t) dydz \quad (27)$$

Next we compute T_1 and T_2 from Equation 14

$$T_1 = \frac{1}{bc} \int_0^b \int_0^c T(0,y,z,t) dydz = F_0(t) \quad (28)$$

where

$$F_0(t) = \frac{1}{bc} \int_0^b \int_0^c f_0(y,z,t) dydz \quad (29)$$

$$T_2 = \frac{1}{bc} \int_0^b \int_0^c T(a,y,z,t) dydz = F_0(t) + aF_1(t) + a^2F_2(t) \quad (30)$$

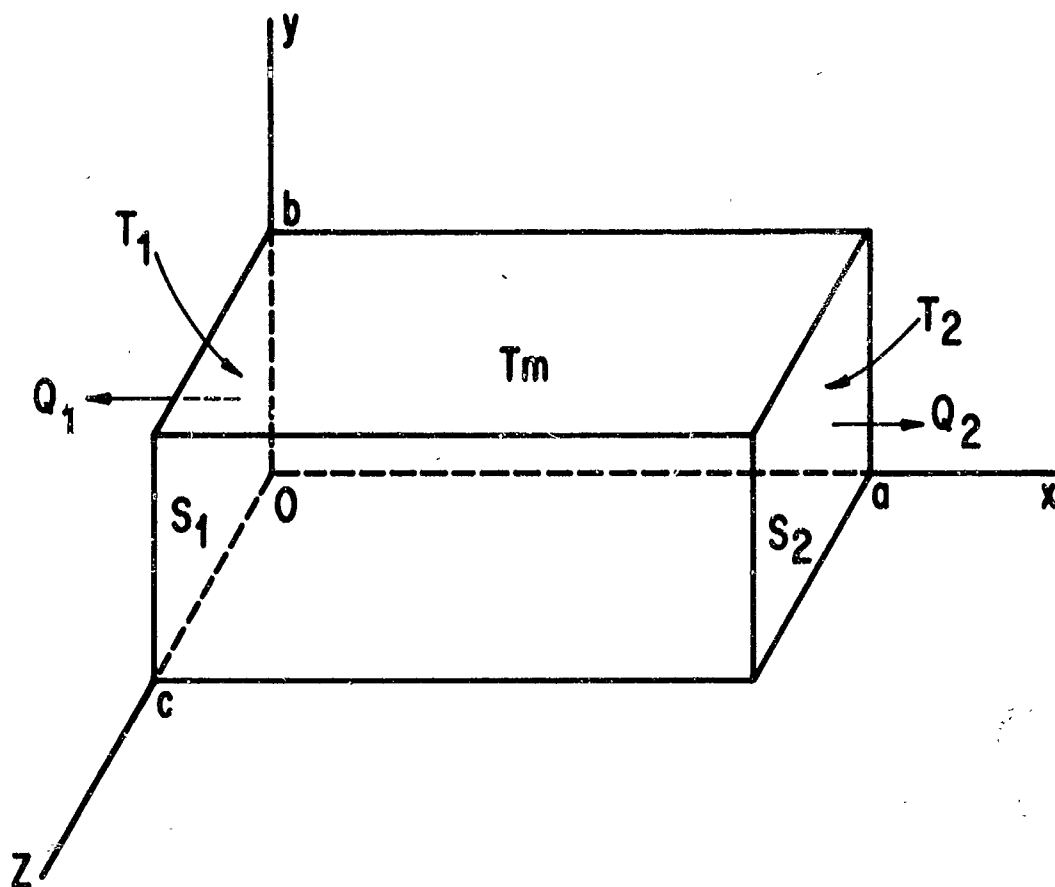


FIGURE 1. RECTILINEAR SOLID: MEAN TEMPERATURES AND BOUNDARY FLUXES.

Finally we find T_m from Equation 7

$$T_m = \frac{1}{abc} \int_0^a \int_0^b \int_0^c T(x,y,z,t) dx dy dz = F_0(t) + \frac{a}{2} F_1(t) + \frac{a^2}{3} F_2(t) \quad (31)$$

We now use the expressions for T_1 , T_2 , and T_m to eliminate the functions F_1 and F_2 from Equations 24 and 26 to find

$$Q_1 = \frac{Kbc}{a} (6T_m - 4T_1 - 2T_2) \quad (32)$$

$$Q_2 = \frac{Kbc}{a} (6T_m - 4T_2 - 2T_1) \quad (33)$$

Similar equations can be written down for the total heat fluxes crossing the two other pairs of faces of the rectilinear solid. These simple expressions for the fluxes can now be substituted into the basic heat balance Equation 11 to yield an ordinary differential equation in the mean temperatures.

$$\begin{aligned} & \frac{Kbc}{a} (12T_m - 6T_1 - 6T_2) + \frac{Kac}{b} (12T_m - 6T_3 - 6T_4) + \frac{Kab}{c} (12T_m - 6T_5 - 6T_6) \\ & - P + C_p abc \frac{dT_m}{dt} = 0 \end{aligned} \quad (34)$$

where T_3 , T_4 , T_5 , and T_6 are the average temperatures over the other faces. The expressions for Q_1 can also be substituted into the boundary conditions, such as those given in Equations 16 and 18. If, for example, face 1 of the rectilinear solid is conductively joined to another rectilinear solid, the boundary condition becomes

$$\frac{Kbc}{a} (6T_m - 4T_1 - 2T_2) + \frac{K'b'c'}{a'} (6T'_m - 4T'_1 - 2T'_2) = 0 \quad (35)$$

Equations of this type will be referred to as joining equations.

Suppose also that face 2 is exposed to sunlight at normal incidence. Then the boundary condition becomes

$$\frac{Kbc}{a} (6T_m - 4T_2 - 2T_1) - bc\epsilon\sigma T_2^4 + bc\alpha s = 0 \quad (36)$$

where α is the solar absorptivity and s is the solar constant, watts cm^{-2} . Four other analogous boundary equations can be written down for the four remaining faces. The six boundary equations, together with the zone heat balance Equation 34, make up the seven equations needed to determine the seven unknown temperatures associated with the zone.

Thin Rectangular Plate

We may now specialize the general case of a rectilinear solid to the case of a thin rectangular plate. Suppose that the dimension c becomes small enough so that the temperature gradient in the z -direction can be neglected. Then the face temperatures T_5 and T_6 become nearly equal to T_m and the quantity

$$\frac{Kab}{c} (12T_m - 6T_5 - 6T_6)$$

approaches indeterminacy as c tends to zero. This term must therefore be

replaced by its limiting value as determined by the boundary conditions.

If the surfaces of the plate are emitting and receiving radiation, the zone heat balance Equation 34 becomes

$$\begin{aligned} & \frac{Kbc}{a} (12T_m - 6T_1 - 6T_2) + \frac{Kac}{b} (12T_m - 6T_3 - 6T_4) + 2abc\epsilon\sigma T_m^4 \\ & - P_5 - P_6 - P + C_p abc \frac{dT_m}{dt} = 0 \end{aligned} \quad (37)$$

where P_5 and P_6 are the powers absorbed by the faces of the plate. In this case boundary conditions need only be written for faces 1, 2, 3, and 4 since T_5 and T_6 have been eliminated.

Rod

Finally, Equation 37 may be specialized to the case of a rod. In this case only the end temperatures and the mean temperature need to be considered and the heat balance equation becomes

$$\frac{KA}{l} (12T_m - 6T_1 - 6T_2) + p l \epsilon \sigma T_m^4 - P + C_p A l \frac{dT_m}{dt} = 0 \quad (38)$$

where A is the cross sectional area of the rod
 l is the length of the rod
 p is the perimeter of the rod.

Annular Cylinder

We consider next the annular cylinder shown in Figure 2. We first restrict the discussion to the case in which there is no azimuthal temperature gradient. In order to obtain the zone formulas for the four outward heat fluxes crossing the faces, we assume that the temperature can be represented sufficiently accurately by

$$T(r,z) = (A + Bz + Cz^2)r^2 + (D + Ez + Fz^2)r + G + Hz + Jz^2 \quad (39)$$

where the coefficients A, B, C, etc., are functions of time.

By methods similar to those employed in analyzing the rectangular solid, we find that

$$Q_1 = \frac{2\pi K l a}{b - a} \left(15T_m - \frac{3b + 5a}{b + a} T_1 - \frac{3b + a}{b + a} T_2 \right) \quad (40)$$

$$Q_2 = \frac{2\pi K l b}{b - a} \left(6T_m - \frac{5b + 3a}{b + a} T_2 - \frac{b + 3a}{b + a} T_1 \right) \quad (41)$$

$$Q_3 = \frac{K\pi(b^2 - a^2)}{l} (6T_m - 4T_3 - 2T_4) \quad (42)$$

$$Q_4 = \frac{K\pi(b^2 - a^2)}{l} (6T_m - 4T_4 - 2T_3) \quad (43)$$

where

- Q_1 is the flux leaving the inner face
- Q_2 is the flux leaving the outer face
- Q_3 is the flux leaving the upper end
- Q_4 is the flux leaving the lower end
- a is the inner radius
- b is the outer radius
- l is the height.

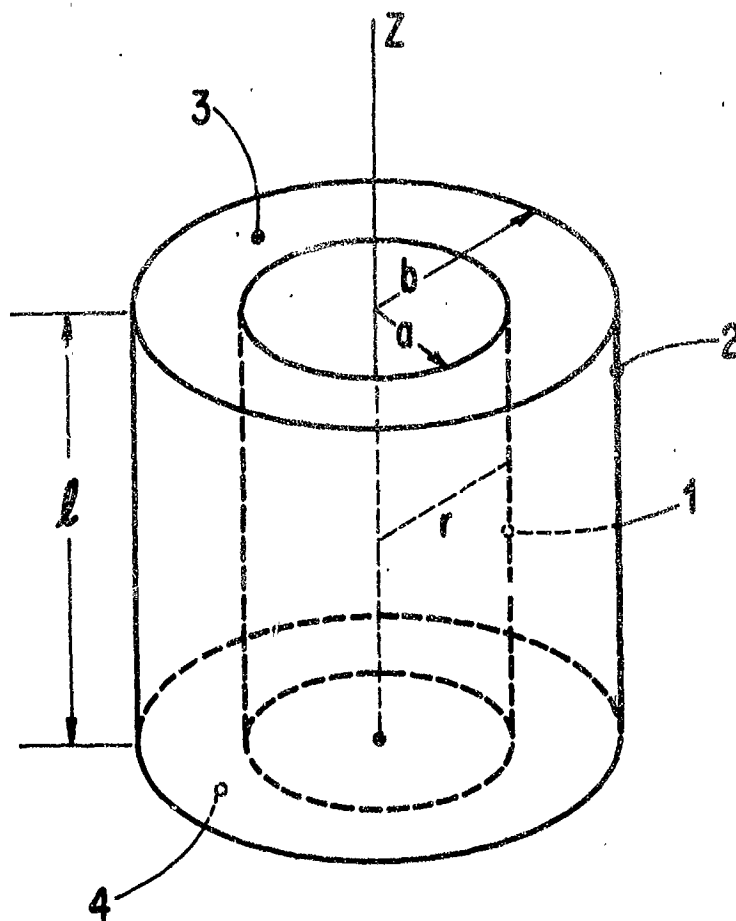


FIGURE 2. ANNULAR CYLINDER: DIMENSIONS AND NUMBERING OF FACES.

These expressions are to be substituted into the zone heat balance Equation 11 and the boundary conditions in the same manner as before. If the ratio a/b approaches unity, the coefficients of the temperatures in Equations 40 and 41 approach the values 6, -4, -2 as they should.

In case the inner radius is zero, the coefficients D, E, and F in Equation 39 must be zero for the temperature to be regular at the origin. In this case Q_1 vanishes and we obtain the expression

$$Q_2 = 8\pi K L (T_m - T_2) \quad (44)$$

Sector of Annular Cylinder

The sector of an annular cylinder is illustrated in Figure 3. The usual correspondence between the face numbers shown and the subscripts will be employed in the discussion that follows. We will now consider gradients in the r , θ , and z directions and assume that the temperature distribution is

$$T(r, \theta, z, t) = A + Br \cos \theta + Cr \sin \theta + Dr^2 \cos 2\theta + Er^2 \quad (45)$$

where the coefficients A, B, C, etc., are functions of t and quadratic functions of z . Inclusion of further items in Equation 45, such as a term in $r^2 \sin 2\theta$, leads to inconsistencies when the fluxes Q_5 and Q_6 are calculated.

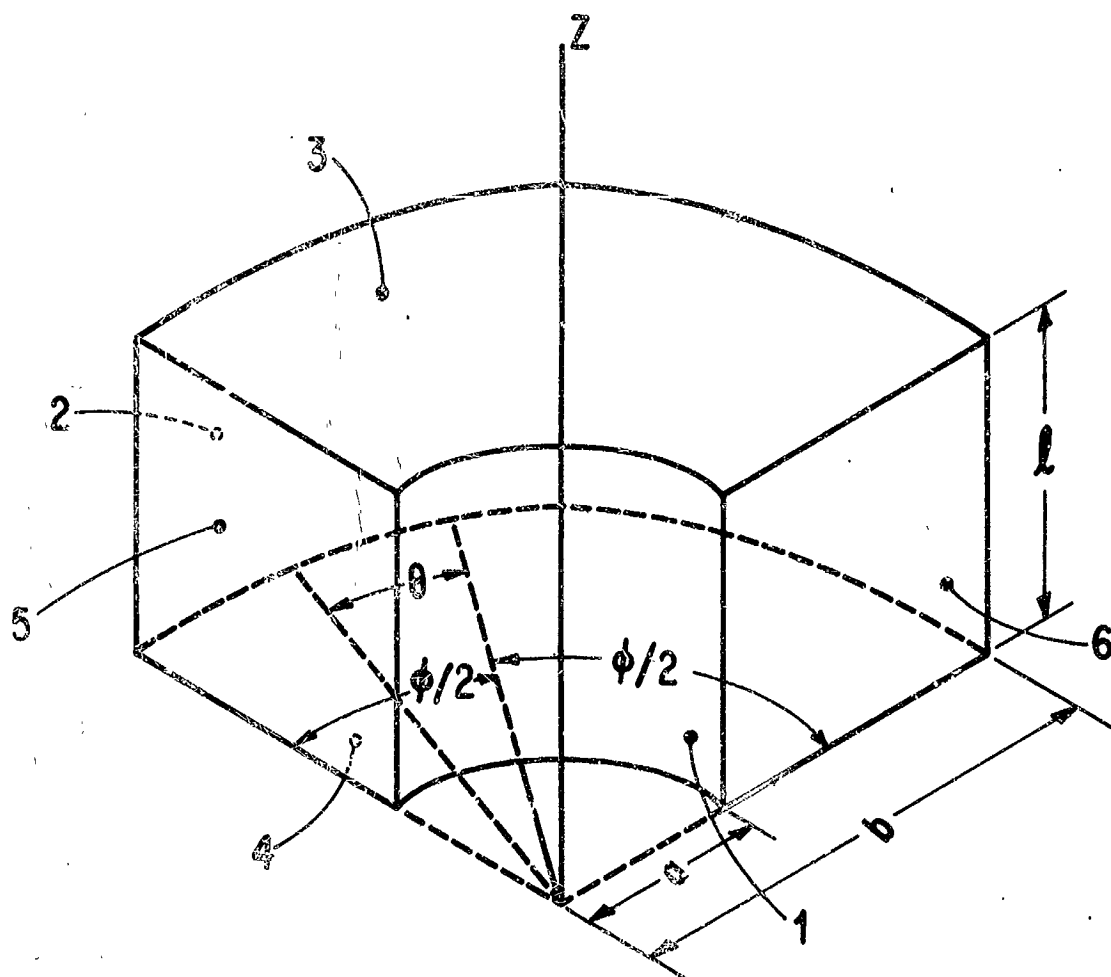


FIGURE 3. SECTOR OF ANNULAR CYLINDER: DIMENSIONS AND NUMBERING OF FACES.

On account of the cylindrical coordinates, a weighted average has to be taken in calculating the mean temperatures T_5 and T_6 . For example,

$$T_5 = \frac{2}{\ell(b^2 - a^2)} \int_0^\ell \int_a^b r T(r, \frac{\phi}{2}, z) dr dz \quad (46)$$

Unless this definition of averaging is used, the average face temperatures T_5 and T_6 will not approach T_m as ϕ tends to zero.

The calculation of Q_5 and Q_6 is straightforward but lengthy so we merely quote the results here

$$Q_5 = K\ell \{2(\gamma - \alpha)T_m - (\beta + \gamma)T_5 - (\gamma - \beta)T_6 + \alpha(T_1 + T_2)\} \quad (47)$$

$$Q_6 = K\ell \{2(\gamma - \alpha)T_m - (\beta + \gamma)T_6 - (\gamma - \beta)T_5 + \alpha(T_1 + T_2)\} \quad (48)$$

where

$$\alpha = \frac{\phi}{2} \left(\frac{b+a}{b-a} \right) \left\{ 4 \left(1 + \frac{\sin\phi - \phi}{\sin\phi - \phi \cos\phi} \right) \left(\frac{b^2 + ab + a^2}{b^2 + a^2} \right) - 3 \right\} \quad (49)$$

$$\beta = \frac{3}{4} \left(\frac{1 + \cos\phi}{\sin\phi} \right) \left(\frac{b^2 - a^2}{b^2 + ab + a^2} \right) \quad (50)$$

$$\gamma = \frac{\phi \sin\phi}{\sin\phi - \phi \cos\phi} \left(\frac{b^2 - a^2}{b^2 + a^2} \right) \quad (51)$$

It will be noted that α and γ tend to infinity as ϕ tends to the root of

$$\phi = \tan\phi$$

which is slightly less than $\frac{3\pi}{2}$. Therefore a sector including an angle much more than $\frac{\pi}{2}$ should be subdivided azimuthally into more than one zone.

The fluxes leaving the other faces are given by formulas essentially the same as those obtained for the complete annular cylinder.

$$Q_1 = \frac{\phi K l a}{b - a} \left(6T_m - \frac{3b + 5a}{b + a} T_1 - \frac{3b + a}{b + a} T_2 \right) \quad (52)$$

$$Q_2 = \frac{\phi K l b}{b - a} \left(6T_m - \frac{5b + 3a}{b + a} T_2 - \frac{b + 3a}{b + a} T_1 \right) \quad (53)$$

$$Q_3 = \frac{\phi K (b^2 - a^2)}{2l} (6T_m - 4T_3 - 2T_4) \quad (54)$$

$$Q_4 = \frac{\phi K (b^2 - a^2)}{2l} (6T_m - 4T_4 - 2T_3) \quad (55)$$

In case the inner radius is zero, as when a circular zone is subdivided azimuthally, then the coefficients in Equation 52 are zero and T_1 is indeterminate. In order to obtain an equation for T_1 , the regularity condition at the origin is invoked. This leads to the condition that

$$\sum_i (6T_{m1} - 3T_1 - 3T_{21}) = 0 \quad (56)$$

where the summation is taken over the subdivisions of the circular zone.

DIFFERENCE EQUATIONS (TIME)

So far we have integrated Equation 1 with respect to the space variables in order to obtain equations of the types (36) and (37) which are difference equations in the space coordinates and may be ordinary differential equations in time. These equations have the general form

$$\sum_j C_{1j} T_j + \sum_j A_{1j} \sigma T_j^4 - P_1(t) + M_1 \frac{dT_1}{dt} = 0 \quad (57)$$

where T_j is a mean temperature, deg K

C_{1j} is a weighted thermal conductance, watts (deg K)⁻¹

A_{1j} is a view area, cm²

$P_1(t)$ is a power, watts

M_1 is a thermal mass, joules (deg K)⁻¹

It is to be noted that Equation 57 typifies both joining or boundary equations and zone heat balance equations. In joining equations, the thermal mass is usually zero. We have discussed in detail the calculation of the weighted conductances C_{1j} . The view areas A_{1j} may be obtained by standard means.⁽¹⁾

We may now integrate Equation 57 with respect to time from t to $t + h$, where h is a small increment in time.

We find

$$\begin{aligned} \sum_j C_{1j} \int_t^{t+h} T_j dt + \sum_j A_{1j} \sigma \int_t^{t+h} T_j^4 dt - \int_t^{t+h} P_1(t) dt \\ + M_1 [T_1(t+h) - T_1(t)] = 0 \end{aligned} \quad (58)$$

In order to obtain approximate expressions for the integrals of T_j and T_j^4 , we use the general linear two-point integration formula:

$$\int_t^{t+h} f(t) dt \approx h[(1 - \alpha) f(t) + \alpha f(t+h)] \quad (59)$$

where f is a function of the time and α is any number lying between zero and one. The choice of the quantity α , which we call the integration parameter, influences the accuracy and stability of the numerical solution of the set of difference equations that has been derived.

Equation 58 becomes

$$h \left\{ \sum_j C_{1j} [(1 - \alpha)T_j(t) + \alpha T_j(t + h)] + \sum_j A_{1j} \sigma [(1 - \alpha)T_j^4(t) + \alpha T_j^4(t + h)] \right\} - \int_t^{t+h} P_j(t) dt + M_1 [T_1(t + h) - T_1(t)] = 0. \quad (60)$$

Equation 60 is one of a set of non-linear simultaneous difference equations. This set of equations can be solved for $T_1(t + h)$ when the values $T_1(t)$ are given.

In order to investigate the factors which determine the optimum value of the integration parameter α , we will compare solutions of Equation 60 with solutions of Equation 57. In order to present this analysis as simply as possible, we investigate the conductive cooling of a lumped thermal mass M with an initial temperature T_0 connected by a conductance C to a sink at zero degrees. If the temperature is T at time t , Equation 57 becomes

$$CT + M \frac{dT}{dt} = 0 \quad (61)$$

The exact solution is

$$T = T_0 e^{-t/\tau} \quad (62)$$

where $\tau = \frac{M}{C}$ (63)

The corresponding difference equation is

$$hC[(1 - \alpha)T(t) + \alpha T(t + h)] + M[T(t + h) - T(t)] = 0 \quad (64)$$

The solution to Equation 64 is

$$T(nh) = T_0 \left(\frac{1 - (1 - \alpha)\frac{h}{\tau}}{1 + \alpha\frac{h}{\tau}} \right)^n \quad (65)$$

The maximum discrepancy between the values of T calculated from (62) and (65) is dependent on α and the ratio h/τ as shown in Figures 4 and 5. The error is given as a percentage of T_0 . The time at which the maximum error occurs is

$$t = h \quad \text{for } h > \tau$$

$$t = \tau \quad \text{for } h < \tau$$

Stability

It will be seen from Equation 65 that if the quantity

$$\beta = \frac{1 - (1 - \alpha)\frac{h}{\tau}}{1 + \alpha\frac{h}{\tau}} \quad (66)$$

becomes greater than unity in magnitude, the temperature $T(nh)$ will tend to infinity instead of zero. This happens when

$$\frac{h}{\tau} > \frac{2}{1 - 2\alpha}, \quad \alpha < \frac{1}{2} \quad (67)$$

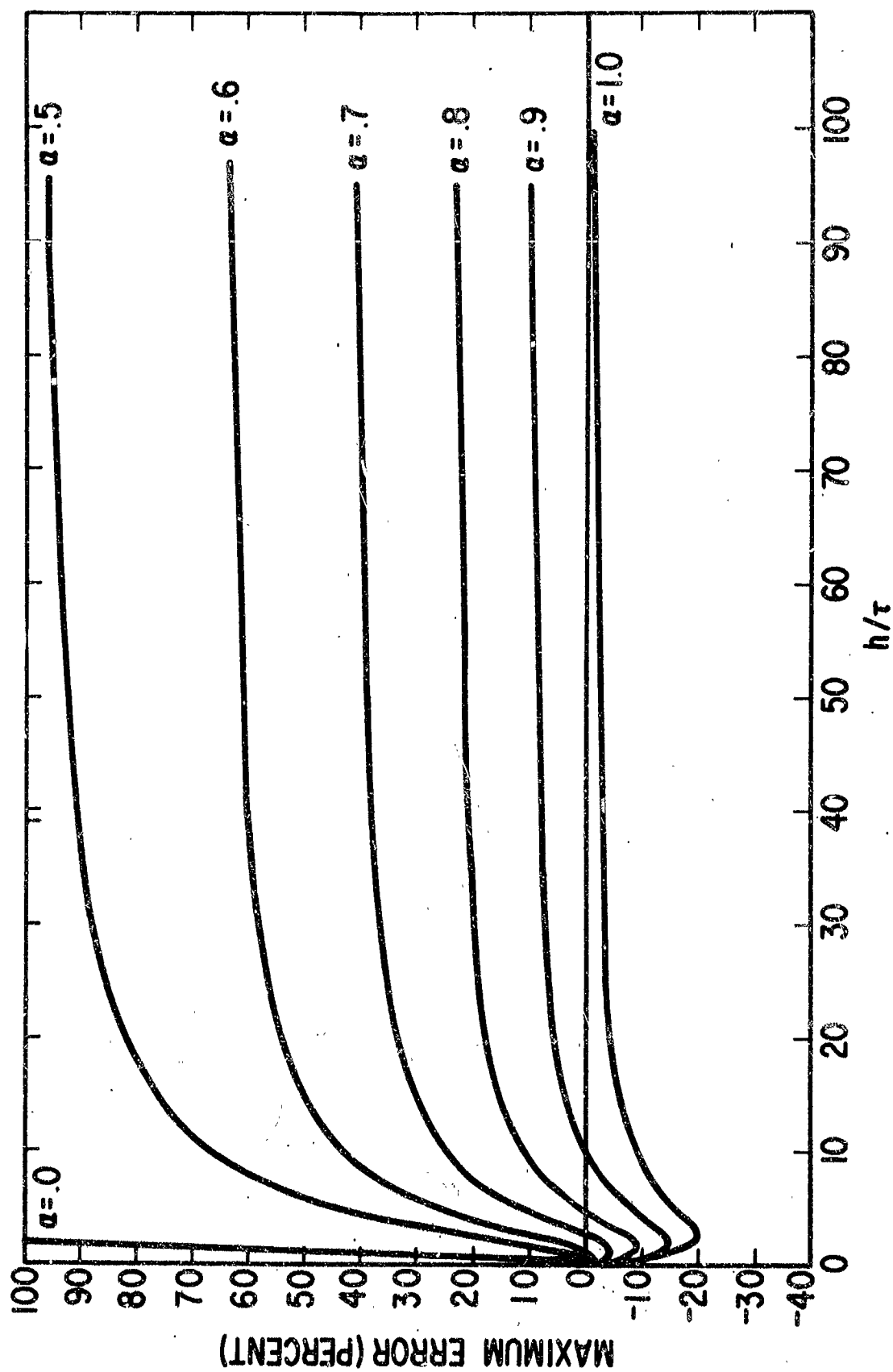


FIGURE 4. MAXIMUM PERCENTAGE ERROR IN TEMPERATURE AS A FUNCTION OF THE RATIO OF TIME INCREMENT TO TIME CONSTANT FOR VARIOUS VALUES OF THE TEMPERATURE DIFFERENCE.

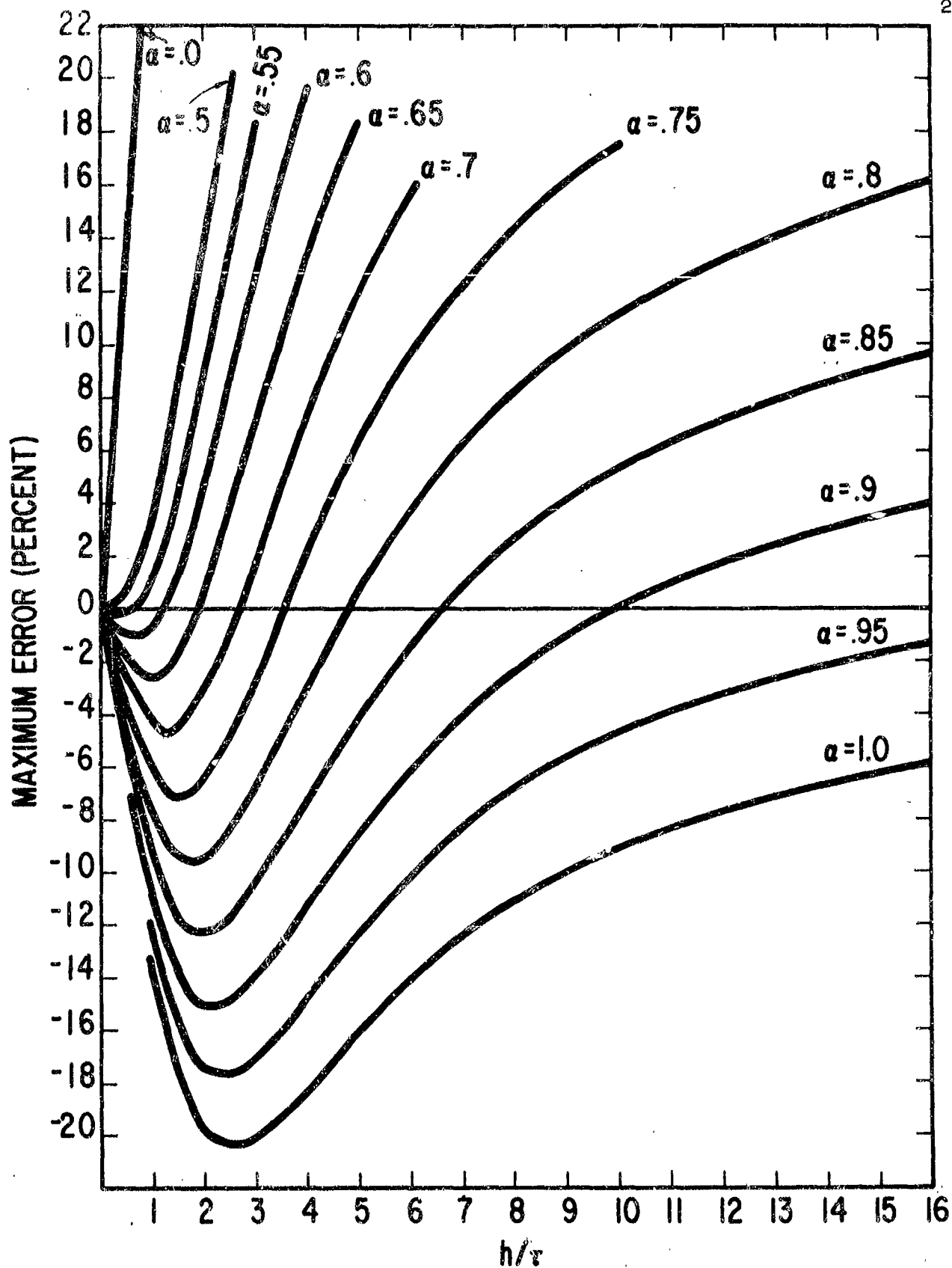


FIGURE 5. DETAILS IN THE NEIGHBORHOOD OF THE ORIGIN OF FIGURE 5.

and never for $\alpha > \frac{1}{2}$. Thus values of α greater than $\frac{1}{2}$ always lead to stable solutions. It is to be noted that the choices of $\alpha = 0$, $\frac{1}{2}$, and 1 lead to the methods of Euler, Crank and Nicholson, and Leasonen, respectively. (2)

In the case of a joining equation which does not contain a term in $\frac{dT}{dt}$, the integration procedure followed still converts the equation into a difference equation. If the joining equation is of the form

$$\Delta Q(t) = C(6T_m - 4T_1 - 2T_2) + C'(6T'_m - 4T'_1 - 2T'_2) = 0$$

as in Equation 35, for example, the numerically integrated equation is

$$h(1 - \alpha)\Delta Q(t) + h\alpha \Delta Q(t + h) = 0. \quad (68)$$

This has the solution

$$\Delta Q(nh) = \left(\frac{\alpha - 1}{\alpha}\right)^n \Delta Q(0) \quad (69)$$

If $\Delta Q(0)$ is not exactly zero, or rounding error creeps in at some stage, the quantity $\Delta Q(nh)$ will tend to infinity unless $\alpha \geq \frac{1}{2}$. In order to secure damping of ΔQ , the integration parameter is always chosen to be greater than one half.

Selection of the Integration Parameter

In order to simplify the discussion of problems involving many zones, we assume the terms in T_j^4 can be linearized. Then the entire set

of differential equations (57) can be written in matrix notation as

$$CT + M \frac{dT}{dt} = P - R \quad (70)$$

where C is a matrix of conductances, including radiative conductances
 T is a vector whose elements are the mean temperatures
 M is a diagonal matrix of the thermal masses
 P is a vector whose elements are the power inputs
 R is a vector of the radiation powers which are the constant terms obtained in the linearization process.

The solution to Equation 70 is

$$T = F E + G(t) \quad (71)$$

where F is a matrix of constants depending on the initial conditions
 E is a vector whose components are e^{-t/τ_k}
 G is a vector representing the particular integral
 τ_k is a time constant.

The time constants τ_k are found by setting the determinant of the matrix $C - \frac{1}{\tau} M$ equal to zero

$$\left| C - \frac{1}{\tau} M \right| = 0. \quad (72)$$

The finite difference equation corresponding to Equation 70 is

$$hC [(1 - \alpha) T(t) + \alpha T(t + h)] + M [T(t + h) - T(t)] = \int_t^{t+h} P dt - hR \quad (73)$$

The solution to Equation 73 is

$$T = F B + H(t) \quad (74)$$

where F is the same matrix introduced in Equation 71

B is a vector with components $(\beta_k)^n$

H is a vector representing the particular sum.

The quantities β_k are found by solving the equation

$$\left| C - \frac{1 - \beta}{h(1 - \alpha + \alpha\beta)} M \right| = 0 \quad (75)$$

Comparison of Equations 75 and 72 shows that

$$\frac{1}{\tau_k} = \frac{1 - \beta_k}{h(1 - \alpha + \alpha\beta_k)} \quad (76)$$

or

$$\beta_k = \frac{1 - (1 - \alpha) \frac{h}{\tau_k}}{1 + \alpha \frac{h}{\tau_k}} \quad (77)$$

If the quantities $\frac{h}{\tau_k}$ vary over a wide range, then it may be seen from Figure 4 that a value of α of about 0.87 minimizes the maximum percentage discrepancy between e^{-nh/τ_k} and $(\beta_k)^n$ over all values of $\frac{h}{\tau_k}$. If the quantities $\frac{h}{\tau_k}$ vary over a small range, however, a smaller value of α gives the minimum error. Figure 6 shows the optimum value of α as a function of the maximum $\frac{h}{\tau_k}$.

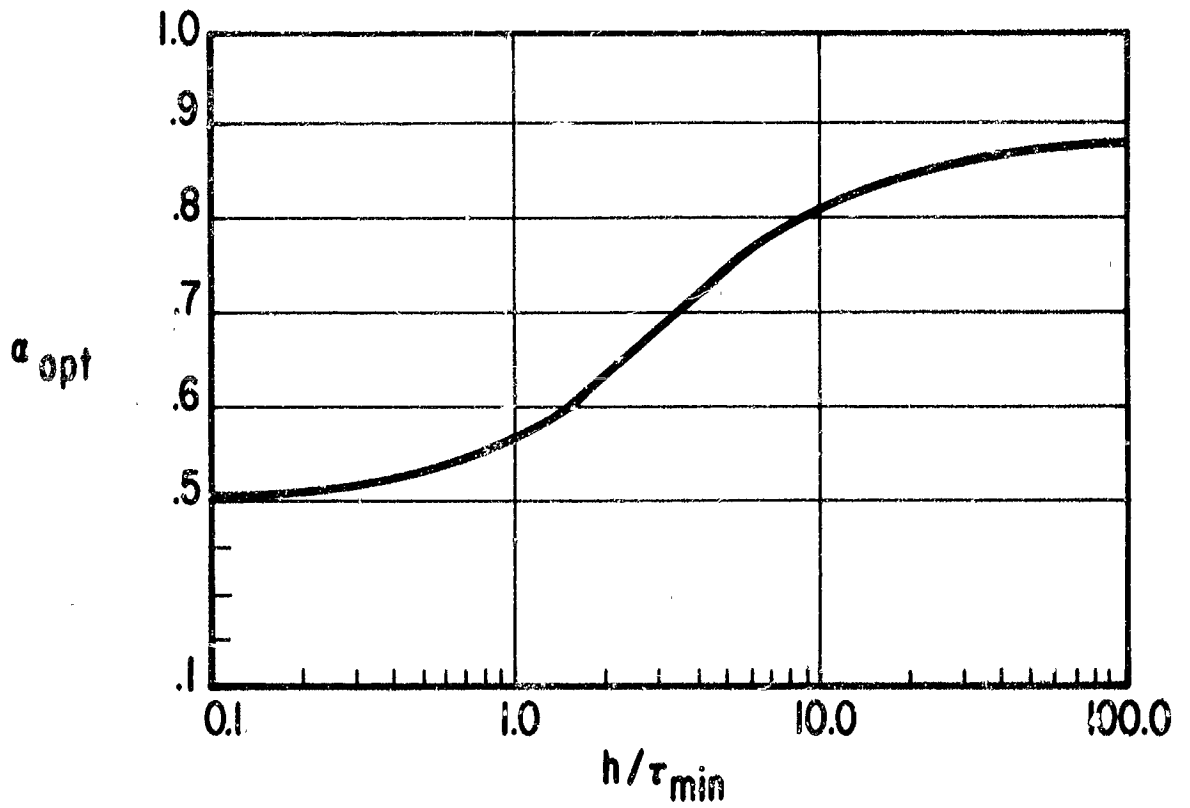


FIGURE 6. THE OPTIMUM VALUE OF THE INTEGRATION PARAMETER AS A FUNCTION OF THE RATIO OF TIME INCREMENT TO SMALLEST TIME CONSTANT.

Selection of the Time Increment

It is shown in Figure 4 that the error introduced by the quadrature in time is reduced by making $\frac{h}{\tau}$ small. However, the time increment h must not be made too small because if a small error ϵ is made at each step in solving the difference equations for $T(t+h)$, this error may be highly magnified. To show this most simply we revert to the case of a single lumped mass. After we introduce the error ϵ , Equation 65 becomes

$$\left. \begin{aligned} T(h) &= T_0 \beta + \epsilon \\ T(2h) &= T_0 \beta^2 + (1 + \beta)\epsilon \\ T(nh) &= T_0 \beta^n + \frac{\epsilon}{1 - \beta} \end{aligned} \right\} \quad (78)$$

We now substitute for β from Equation 66 and find

$$T(nh) = T_0 \beta^n + \left(\frac{\tau}{h} + \alpha \right) \epsilon \quad (79)$$

It is seen from Equation 79 that a correlated error will be magnified by the factor $\left(\frac{\tau}{h} + \alpha \right)$ which will be large if $\frac{h}{\tau}$ is small. A generalization of the argument shows that magnification factors $\left(\frac{\tau_k}{h} + \alpha \right)$ occur in multizone problems.

SOME SIMPLE EXAMPLES

Bar Losing Heat

As a first example to illustrate the application of the method of zones to a steady state problem, we consider the case of a bar losing heat along its length at a rate determined by a heat transfer coefficient H . The ends of the bar are held at fixed temperatures of 0° and T_1 . By the method of zones the heat balance equation, as in Equation 38, is

$$\frac{KA}{l} (12T_m - 6T_1) + pHlT_m = 0 \quad (80)$$

where A is the cross sectional area of the bar

l is the length of the bar

p is the perimeter of the bar.

Equation 80 may be solved immediately to give

$$T_m = \frac{6KAT_1}{12KA + pHl^2} \quad (81)$$

To compare this result with the exact result and the result obtained by the nodal method, it is desirable to calculate the center temperature T_c . Since the temperature of the zone is assumed to be parabolic, the center temperature is given by the expression

$$T_c = \frac{6T_m - T_1}{4} = \frac{24 - (\frac{l}{\delta})^2}{48 + 4(\frac{l}{\delta})^2} T_1 \quad (82)$$

where

$$\delta = \sqrt{\frac{KA}{pH}}$$

The exact result is

$$T_c = \frac{\sinh \frac{l}{2\delta}}{\sinh \frac{l}{\delta}} T_1 \quad (84)$$

By the nodal method the equation for the center temperature is

$$\frac{2KA}{l} T_c + \frac{2KA}{l} (T_c - T_1) + p h l T_c = 0 \quad (85)$$

whence

$$T_c = \frac{2}{4 + \left(\frac{l}{\delta}\right)^2} T_1 \quad (86)$$

Some numerical results for comparison are given in Table I. One sees that the method of zones gives quite good results when $\frac{l}{\delta}$ is two or less, and

TABLE I

T_c/T_1			
l/δ	Exact	Method of Zones	Nodal Method
1	0.436	0.443	0.400
2	0.324	0.312	0.250
3	0.213	0.178	0.154

that the error of the nodal method is about five times that of the method of zones in this range.

Triangular Tube in Sunlight

Figure 7 shows a cross section of a long triangular tube in sunlight. We will set up the zone heat balance equations for three zones, numbered 2, 4, and 6 in the diagram, and the joining equations for the edges, numbered 1, 3, and 5, in order to illustrate the application of the method to a multizone problem. We consider a unit length of the tube which is taken to be black both inside and out in the infrared.

The following equations describe the system:

$$\frac{K_6 \delta_6}{l_6} (6T_6 - 4T_1 - 2T_5) + \frac{K_2 \delta_2}{l_2} (6T_2 - 4T_1 - 2T_3) = 0 \quad (87)$$

$$\begin{aligned} \frac{K_2 \delta_2}{l_2} (12T_2 - 6T_1 - 6T_3) + 2l_2 \sigma T_2^4 - l_{26} \sigma T_6^4 - l_{24} \sigma T_4^4 \\ - \alpha_2 l_2 s \cos \theta_2 + M_2 \frac{dT_2}{dt} = 0 \end{aligned} \quad (88)$$

$$\frac{K_2 \delta_2}{l_2} (6T_2 - 4T_3 - 2T_1) + \frac{K_4 \delta_4}{l_4} (6T_4 - 4T_3 - 2T_5) = 0 \quad (89)$$

$$\begin{aligned} \frac{K_4 \delta_4}{l_4} (12T_4 - 6T_3 - 6T_5) + 2l_4 \sigma T_4^4 - l_{24} \sigma T_2^4 - l_{46} \sigma T_6^4 \\ - \alpha_4 l_4 s \cos \theta_4 + M_4 \frac{dT_4}{dt} = 0 \end{aligned} \quad (90)$$

$$\frac{K_4 \delta_4}{l_4} (6T_4 - 4T_5 - 2T_3) + \frac{K_6 \delta_6}{l_6} (6T_6 - 4T_5 - 2T_1) = 0 \quad (91)$$

$$\begin{aligned}
& \frac{K_6 \delta_6}{l_6} (12T_6 - 6T_1 - 6T_5) + 2l_6 \sigma T_6^4 - l_{26} \sigma T_2^4 - l_{46} \sigma T_4^4 \\
& + M_6 \frac{dT_6}{dt} = 0
\end{aligned} \tag{92}$$

In these equations

δ_i is the thickness of zone i

K_i is the conductivity of zone i

l_i is the width of zone i

l_{ij} is the view area per unit length between zones i and j

α_i is solar absorptivity of zone i

s is the solar constant

θ_i is the angle of incidence of the sunlight on zone i

M_i is thermal mass of zone i

T_2 , T_4 , and T_6 are the mean temperatures of zones 2, 4, and 6,
respectively

T_1 , T_3 , and T_5 are the temperatures of the vertices of the
triangles as shown in Figure 7

Equations 87, 89, and 91 are the joining equations; while Equations 88, 90, and 92 are the zone heat balance equations. It will be observed that six equations in six unknowns are produced so that the problem is properly defined and soluble if the temperatures T_2 , T_4 , and T_6 are given initially. The initial values of temperatures T_1 , T_3 , and T_5 are found from Equations 87, 89, and 91 solved simultaneously. The six equations may now be integrated numerically (without linearization of T_i^4) as described earlier.

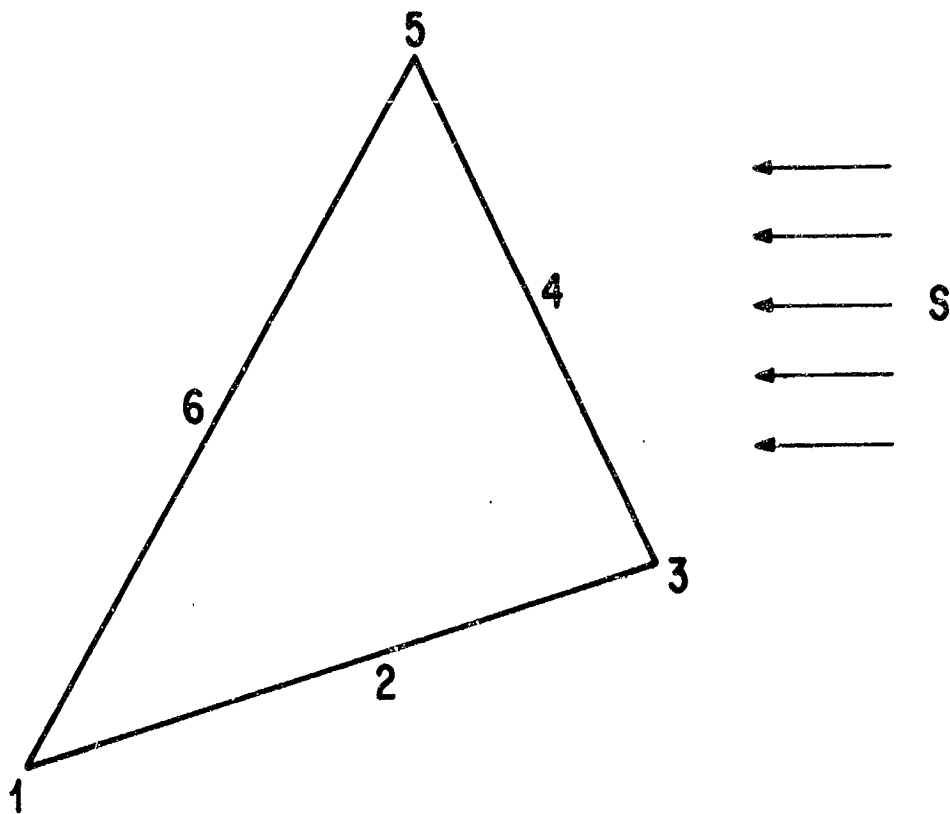


FIGURE 7. CROSS SECTION OF A TRIANGULAR TUBE IN SUNLIGHT
SHOWING NUMBERING OF FACES AND EDGES.

Cooling of a Slab

Consider an infinite slab of thickness l thermally insulated over one face (at $x = 0$) and with the temperature of the other face (at $x = l$) held at zero degrees. The initial temperature of the slab is taken to be T_1 . We shall derive the solution of this transient problem by the method of zones using only one zone and compare this result with the exact answer.

To do the problem by the method of zones, let T_0 be the temperature of the boundary at $x = 0$ and T_m be the mean temperature of the slab. Then the adiabatic boundary condition is expressed by the equation

$$6T_m - 4T_0 = 0 \quad (93)$$

The overall heat balance equation is

$$\frac{K}{l} (12T_m - 6T_0) + C\rho l \frac{dT_m}{dt} = 0 \quad (94)$$

In order to investigate the effect of the use of a single zone on the transient behavior of the temperature, we eliminate T_0 and integrate to obtain

$$T_m = T_1 e^{-\frac{3\kappa t}{l^2}} \quad (95)$$

where

$$\kappa = \frac{K}{C\rho}, \text{ cm}^2 \text{ sec}^{-1}.$$

The exact answer for the temperature distribution is

$$T(x,t) = \frac{4}{\pi} T_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos \frac{(2n+1)\pi x}{2l} e^{-\frac{(2n+1)^2 \pi^2 \kappa t}{4l^2}} \quad (96)$$

The mean temperature is

$$T_m = \frac{8}{\pi^2} T_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} e^{-\frac{(2n+1)^2 \pi^2 \kappa t}{4l^2}} \quad (97)$$

Figure 8 shows plots of the exact and approximate values of $\frac{T_m}{T_1}$ as functions of $\kappa t/l^2$. The difference, $\frac{\delta T_m}{T_1}$, between the two functions is shown in Figure 9. It will be noted that the error is less than about 2% of T_1 for values of $\frac{\kappa t}{l^2}$ greater than 0.3. The large error at the beginning, approximately 11% of T_1 , is due to the fact that in the initial stages of the transient the temperature distribution is very far from parabolic. In order to obtain accurate results in the early stage, two or more zones should be used.

CRITERIA FOR CHOICE OF ZONE SIZE

After the heat balance equations have been written one still enjoys freedom of choice of the time interval and the integration parameter. However, no such freedom remains in the choice of the zone sizes. If the zones are made too large, excessive error in the temperature distribution will result. On the other hand, if the zones are too small, excessive labor is required to set up and solve the problem. Therefore

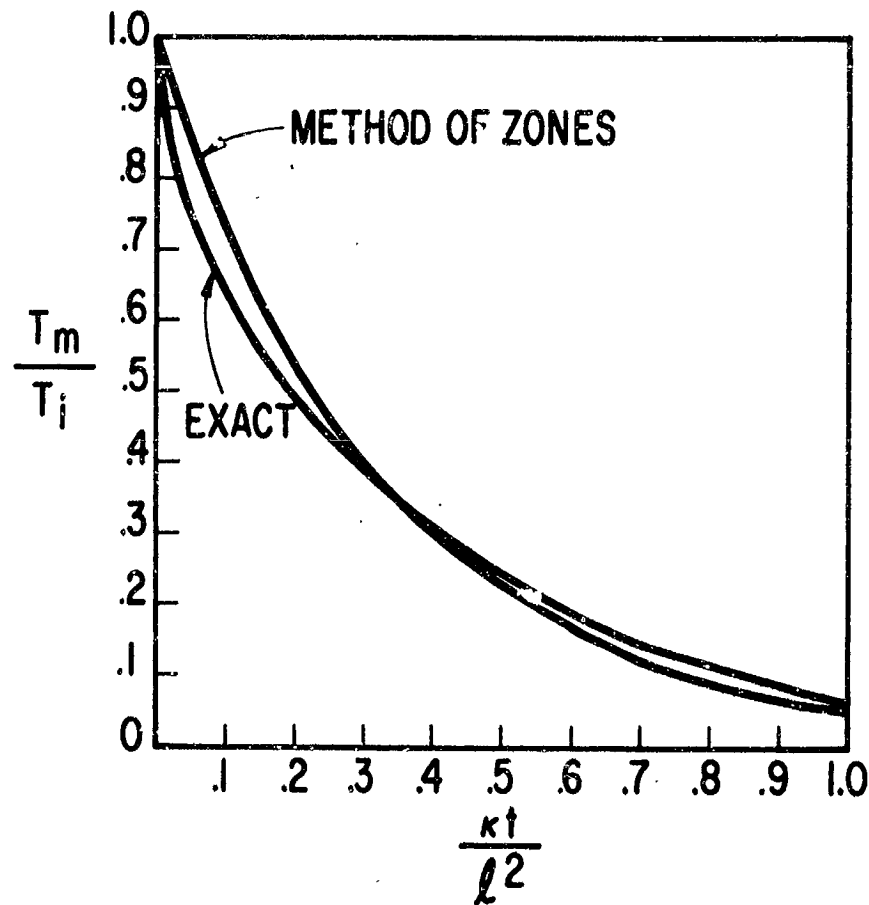


FIGURE 8. COMPARISON OF THE METHOD OF ZONES CALCULATION WITH THE EXACT SOLUTION IN THE PROBLEM OF THE COOLING OF A SLAB. THE RATIO OF MEAN TEMPERATURE T_m TO INITIAL TEMPERATURE T_i IS PLOTTED AGAINST kt/l^2 A DIMENSIONLESS VARIABLE PROPORTIONAL TO TIME WHERE κ IS THE THERMAL DIFFUSIVITY AND l IS THE THICKNESS OF THE SLAB.

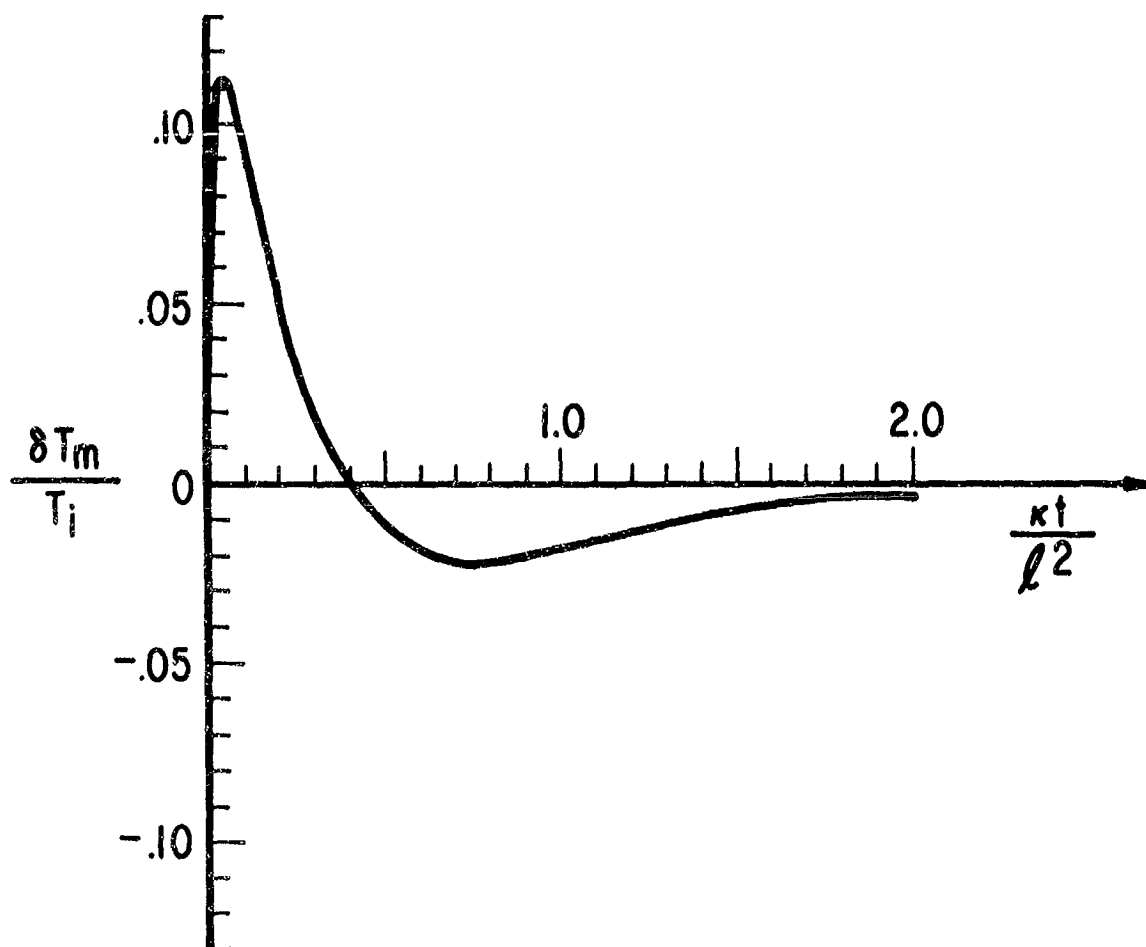


FIGURE 9. PLOT OF THE DIFFERENCE OF THE TWO GRAPHS OF FIGURE 8.

it is vital to develop criteria which permit one to choose zones which are of the maximum size consistent with the tolerable error.

The principle used to derive these criteria is that the zone size must be chosen so that the parabolic approximation is valid at all times of interest. An immediate consequence of this principle is that if the temperature distribution in a region is strongly S-shaped, the region must be broken into at least two zones. For example, if a cylindrical satellite is rotating about its axis in sunlight, a minimum of three, and preferably four, azimuthal zones must be used because of the approximately sinusoidal distribution of temperature.

Many exact solutions to heat transfer problems consist of a series of exponentials and trigonometric functions. In order to obtain the maximum size of a zone, we calculate the interval over which the exponential or trigonometric function is well represented by a parabola. For example, consider the case of a semi-infinite rod extending from $x = 0$ to $x = \infty$ and losing heat along its length. Suppose that the temperature at the origin is modulated sinusoidally with an amplitude T_0 and angular frequency ω . The differential equation is

$$-KA \frac{\partial^2 T}{\partial x^2} + h\rho T + C\rho A \frac{\partial T}{\partial t} = 0 \quad (98)$$

where K is the conductivity

A is the cross-sectional area

h is the heat transfer coefficient

p is the perimeter

C_p is the heat capacity per unit volume

T is the temperature relative to ambient.

When steady state is reached the temperature is

$$T = T_0 e^{-\frac{x}{\delta}} \cos(\omega t - \frac{2\pi x}{\lambda}) \quad (99)$$

where

$$\delta = \sqrt{\frac{2}{\left(\frac{hp}{KA}\right)^2 + \left(\frac{\omega}{\kappa}\right)^2 + \frac{hp}{KA}}} \quad (100)$$

$$\lambda = 2\pi \sqrt{\frac{2}{\left(\frac{hp}{KA}\right)^2 + \left(\frac{\omega}{\kappa}\right)^2 - \frac{hp}{KA}}} \quad (101)$$

$$\kappa = \frac{K}{C_p}$$

The quantity δ may be called the temperature decay length and λ is the wavelength of the damped temperature wave in the rod. The zone size may be of the order of δ to 3δ or $\frac{\lambda}{4}$, whichever is smaller. It will be noted that δ is less than $\lambda/4$.

For $\omega = 0$, the decay length δ always governs (because $\lambda = \infty$) and is given by

$$\delta = \sqrt{\frac{KA}{hp}} \quad (102)$$

In case of a high frequency modulation

$$\lambda = 2\pi\delta = 2\pi\sqrt{\frac{2\kappa}{\omega}} \quad (103)$$

The zone size should therefore be chosen to be about $\frac{\pi}{2}\sqrt{\frac{2\kappa}{\omega}}$, that is, a quarter wavelength.

Another criterion is obtained by examination of Figure 9, which shows the error in a transient problem. For an error in the mean temperature of less than about 2% of the total temperature swing, sufficient time must elapse so that

$$t \geq \frac{0.3l^2}{\kappa} \quad (104)$$

where l is the size of the zone. In other words, if t_0 is the earliest time of interest, the zone size l should be chosen so that

$$l \leq 1.8\sqrt{\kappa t_0} \quad (105)$$

An additional type of restriction on zone size has already been given in Inequality 15. This is the condition that

$$\sigma \overline{T}^4 \leq \sigma \overline{T}_1^4$$

over radiating surfaces. Condition 15 can be put in the simple form

$$l \ll \frac{T}{\left| \frac{\partial T}{\partial x} \right|} \quad (106)$$

NUMERICAL SOLUTION OF THE DIFFERENCE EQUATIONS

While the modified Gauss-Seidel procedure used for solving the difference Equations 60 is well known, it will be briefly described here for the sake of completeness and also because the method has been found to work well in practice on a large number of complicated examples.

In order to calculate $T_1(t + h)$, iteration with an acceleration factor is used. The values of $T_1(t)$ are used as a starting approximation to $T_1(t + h)$. New approximations are obtained successively by solving the equations

$$f_{01} + f_{11} T_1^*(t + h) + f_{41} (T_1^*(t + h))^4 = 0 \quad (107)$$

$$\begin{aligned} \text{where } f_{01} = & h \sum_j C_{1j} (1 - \alpha) T_1(t) + h \sum_{j \neq 1} C_{1j} \alpha T_j(t + h) \\ & + h \sum_j A_{1j} (1 - \alpha) \sigma T_j^4(t) + h \sum_{j \neq 1} A_{1j} \alpha \sigma T_j^4(t + h) \\ & - \int_t^{t+h} P_1 dt - M_1 T_1(t) \end{aligned} \quad (108)$$

$$f_{11} = h C_{11} \alpha + M_1 \quad (109)$$

$$f_{41} = h A_{11} \alpha \sigma \quad (110)$$

When the root $T_1^*(t + h)$ has been found, the old value of $T_1(t + h)$ is replaced by

$$T_i(t + h) + \gamma [T_i^*(t + h) - T_i(t + h)] \quad (111)$$

and the next equation is set up and solved. The acceleration factor γ is generally chosen to be about 1.6 for optimum results.

Convergence of the method is generally good unless two or more temperatures are very tightly coupled to each other and weakly coupled to surrounding temperatures. This situation is to be avoided by replacing two such relatively tightly coupled temperatures by a single temperature.

CHECKING PROCEDURES

It is a laborious and tedious task to set up all of the numerical coefficients used in the equations describing the heat balance and joining conditions of many zones in a complicated problem, so that errors frequently creep in. Many of these errors can be detected by checks based on the principles of reciprocity and the conservation of energy. In general, the rows and columns of the matrices of conductances and view areas should add up to zero. This will be so if the equations are written in full exactly as prescribed by the method of zones with all diagonal elements positive. While it would make no difference to the final answer if various equations were multiplied by different constants, such manipulation would make it impossible to check the column sums. Certain complications generally do arise, however. In the first place, it frequently happens that certain areas of the system are radiating to

empty space, so that the view area for the radiation from empty space is omitted.

When any part of the system (e.g., a boundary) has a fixed, pre-assigned temperature T , the equation for this temperature is simply

$$\frac{dT}{dt} = 0 \quad (112)$$

It will be seen that the conductances and view areas of the rest of the system to this part are omitted, but not conversely, so that non-zero column sums will occur. As long as all the omitted conductances and areas are known, however, the non-zero sums can be checked.

Non-zero column sums also arise when circular zones, subdivided azimuthally, are present because the condition on the center temperature is the regularity condition and not a flux condition. Here again, however, the amount of the column sum can be predicted and thus the column can still be checked.

Another check that can be made is on the overall heat energy balance of the entire system. Five terms enter the energy balance equation

$$\sum_i M_i T_i(t) = \sum_i M_i T_i(0) + \sum_i \int_0^t P_i dt - \sum_i A_i \sigma \int_0^t T_i^4 dt - \sum_i C_i \int_0^t T_i dt \quad (113)$$

where $\sum_i M_i T_i(t)$ is the energy stored at time t

$\sum_i M_i T_i(0)$ is the initial stored energy

$\sum_i \int_0^t P_i dt$ is the energy received

$\sum_i A_i \sigma \int_0^t T_i^4 dt$ is the net energy lost by radiation

$\sum_i C_i \int_0^t T_i dt$ is the net energy lost by conduction.

In Equation 113 the quantities A_i and C_i are simply the areas and conductances omitted in the matrices of coefficients. It is to be noted that the integrations of T_i and T_i^4 should be carried out with the same integration parameter α used for the integration of the individual heat balance equations. This check can be applied at any time t after the start of the calculation.

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References

- ¹ W. H. McAdams, Heat Transmission (McGraw-Hill Book Company, Inc., New York, 1942).
- ² R. D. Richtmyer, Difference Methods for Initial-Value Problems (Interscience Publishers, Inc., New York, 1957).